

CONVERGENCE OF LINEAR FUNCTIONALS OF THE GRENANDER ESTIMATOR UNDER MISSPECIFICATION

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ABSTRACT. Under the assumption that the true density is decreasing, it is well known that the Grenander estimator converges at rate $n^{-1/3}$ if the true density is curved (Prakasa Rao, 1969) and at rate $n^{-1/2}$ if the density is flat (Groeneboom and Pyke, 1983; Carolan and Dykstra, 1999). In the case that the true density is misspecified, the results of Patilea (2001) tell us that the global convergence rate is of order $n^{-1/3}$ in Hellinger distance. Here, we show that the local convergence rate is $n^{-1/2}$ at a point where the density is misspecified. This is not in contradiction with the results of Patilea (2001): the global convergence rate simply comes from locally curved *well-specified* regions. Furthermore, we study global convergence under misspecification by considering linear functionals. The rate of convergence is $n^{-1/2}$ and we show that the limit is made up of two independent terms: a mean-zero Gaussian term and a second term (with non-zero mean) which is present only if the density has well-specified locally flat regions.

1. INTRODUCTION

Shape-constrained nonparametric maximum likelihood estimators provide an intriguing option to kernel-based density estimators. For example, one can compare the standard histogram with the Grenander estimator for a decreasing density. Rules exist to pick the bandwidth (or bin width) for the histogram to attain optimal convergence rates, cf. Wasserman (2006). On the other hand, the Grenander estimator gives a piecewise constant density, or histogram, but the bin widths are now chosen completely automatically by the estimator. Furthermore, the bin widths selected by the Grenander estimator are naturally locally adaptive (Birgé, 1987). Similar comparisons can also be made between the log-concave nonparametric MLE and the kernel density estimator with, say, the Gaussian kernel.

The Grenander estimator was first introduced in Grenander (1956) and has been considered extensively in the literature since then. A recent review of the history of the problem appears in Durot et al. (2012). The latter paper establishes that the Grenander estimator converges to a true strictly decreasing density at a rate of $(n/\log n)^{-1/3}$ in the L_∞ norm. Other rates have also been derived over the years, most notably, convergence at a point at a rate of $n^{-1/3}$ if the true density is locally

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strictly decreasing (Prakasa Rao, 1969; Groeneboom, 1985) and at a rate of $n^{-1/2}$ if the true density is locally flat (Groeneboom, 1983; Carolan and Dykstra, 1999).

As noted in Cule et al. (2010); Dümbgen et al. (2011) the “success story” of maximum likelihood estimators is their robustness. Namely, let \mathcal{F} denote the space of decreasing densities on \mathbb{R}_+ . Next, let f_0 denote the true density and \hat{f}_0 denote the density closest to f_0 in the Kullback-Leibler sense. That is,

$$\hat{f}_0 = \operatorname{argmin}_{g \in \mathcal{F}} \int_0^\infty f_0(x) \log \frac{f_0(x)}{g(x)} dx.$$

We will call the density \hat{f}_0 the KL projection density of f_0 , or the KL projection for short. Note that if $f_0 \in \mathcal{F}$ then $\hat{f}_0 = f_0$. Patilea (2001) showed that the density \hat{f}_0 exists and is unique, and that the Grenander estimator converges to \hat{f}_0 when the observed samples come from the true density f_0 , regardless if $f_0 \in \mathcal{F}$. Similar results were proved for the log-concave maximum likelihood estimator in Cule and Samworth (2010); Cule et al. (2010); Dümbgen et al. (2011); Balabdaoui et al. (2012).

Let \hat{f}_n denote the Grenander estimator of a decreasing density. We show here that at a point where the density is misspecified the rate of convergence of \hat{f}_n to \hat{f}_0 is $n^{-1/2}$, and we also identify the limiting distribution. This is not in contradiction with the results of Patilea (2001): the slower $n^{-1/3}$ global convergence rate simply comes from locally curved *well-specified* regions. To be more specific, if the density f_0 is misspecified at a point, then \hat{F}_0 must be linear (and \hat{f}_0 is flat), and in regions where \hat{f}_0 is flat the rate of convergence is $n^{-1/2}$. In fact, the $n^{-1/2}$ rate holds at all flat regions of \hat{f}_0 , irrespective of whether these are miss- or well-specified. The complete result is given in Section 2, where some properties of \hat{f}_0 are also discussed.

Next, we consider convergence of linear functionals. Let

$$\hat{\mu}_0(g) = \int_0^\infty g(x) \hat{f}_0(x) dx \quad \text{and} \quad \hat{\mu}_n(g) = \int_0^\infty g(x) \hat{f}_n(x) dx. \quad (1.1)$$

In Section 3 we show that $n^{1/2}(\hat{\mu}_n(g) - \hat{\mu}_0(g)) = O_p(1)$, and we again identify the limiting distribution. Notably, the limit is made up of two *independent* terms: a mean-zero Gaussian term and a second term with non-zero mean. Furthermore, the second term is present only if the density has well-specified locally flat regions. Our results apply to a wide range of KL projections with both strictly curved and flat regions. The work in the strictly curved case follows from the rates of convergence of $\hat{F}_n(y) = \int_0^y \hat{f}_n(y) dy$ to the empirical distribution function established in Kiefer and Wolfowitz (1976). However, as mentioned above, these are only the well-specified regions of \hat{f}_0 . A related work here is that of Kulikov and Lopuhaä (2008), who consider functionals in the strictly curved case but at the distribution function level.

In Section 4 we go beyond the linear setting, and consider convergence of the entropy functional in the misspecified case. The limit in this case is Gaussian, irrespective of at the local properties of \hat{f}_0 . Most proofs appear in Section 5 and some technical

details are left to the Appendix. Throughout, our results are illustrated by reproducible simulations. Code for these is available online at www.math.yorku.ca/~hkj/.

To our best knowledge, previous work on asymptotics under misspecification in the shape-constrained context is limited to the rates established in van de Geer (2000) and Patilea (2001), as well as the more recent results of Balabdaoui et al. (2012). In Balabdaoui et al. (2012), the pointwise asymptotic distribution under misspecification was derived for the log-concave probability mass function.

The implications of the new results obtained here are as follows. First, we now understand that \hat{f}_0 will be made up of local well-specified and misspecified regions, and that the rate of convergence in the misspecified regions is always $n^{-1/2}$. We conjecture that this type of behaviour will be seen in other situations, such as the log-concave setting for $d = 1$. That is, the rate of convergence in misspecified regions will be $n^{-1/2}$ whereas in well-specified regions the rate of convergence will depend on whether locally the density lies on the boundary or the interior of the underlying space. In the log-concave $d = 1$ case, this “interior” rate is known to be $n^{-2/5}$ (Balabdaoui et al., 2009). A generally accepted conjecture is that here, the “boundary” rate will again be $n^{-1/2}$. The interesting case of $d > 1$ is more mysterious though, as the relationship between the slower boundary points and faster interior points is harder to identify. Thus, our results provide new insight on the limiting distribution even in the completely well-specified context.

Secondly, we show that linear functionals (as well as the non-linear entropy functional) converge at rate $n^{-1/2}$, and we also conjecture that this behaviour will continue to hold for other shape constraints. Let $\mu_0(g) = \int_0^\infty g(x)f_0(x)dx$. Our results show that

$$\sqrt{n}(\hat{\mu}_n(g) - \mu_0(g)) = O_p(1) + \sqrt{n}(\hat{\mu}_0(g) - \mu_0(g)). \quad (1.2)$$

Therefore, global rates of *divergence* are $n^{+1/2}$ for linear functionals in the misspecified case. A similar statement also holds for the entropy functional, and here the random $O_p(1)$ term is always Gaussian. Such results are well-understood in parametric settings, and are key in power calculations. The exact conditions necessary for (1.2) to hold are given in Section 3 for $\mu_0(g)$ and in Section 4 for the entropy. Our work can also be easily extended to locally misspecified settings such as those studied in Le Cam (1960).

Lastly, the fact that the limiting distribution of the linear functional $\hat{\mu}_n(g)$ depends on properties of \hat{f}_0 , whereas the limiting distribution of the entropy functional is always Gaussian, makes the entropy functional potentially more appealing in terms of testing procedures. A hypothesis test based on a modified version of the entropy functional was considered in Cule et al. (2010) whereas the “trace test” of Chen and Samworth (2011) depends on a nearly linear functional, the variance. Both, however, are developed in the context of log-concavity, and it would be of great interest to extend the results presented here to that setting, particularly for higher dimensions.

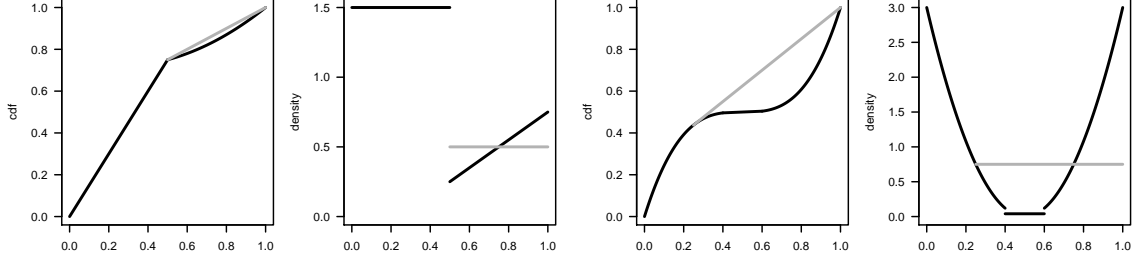


FIGURE 1. Two examples of f_0 and $\hat{f}_0 = \text{gren}(F_0)$. The two left panels show the cdf and density for example (2.1) while the two right panels show the cdf and density for example (2.2). F_0 (resp. f_0) is shown in black, and \hat{F}_0 (resp. \hat{f}_0) is shown in gray, but only if different from the truth (namely F_0 and f_0 respectively).

2. THE KULLBACK-LEIBLER PROJECTION AND POINTWISE CONVERGENCE UNDER MISSPECIFICATION

Properties of the KL projection onto the space of log-concave densities were studied in Dümbgen et al. (2011). When projecting onto the space of decreasing densities, the behaviour is a little easier to characterize.

Theorem 2.1. (*Patilea, 1997, 2001*) *Let f_0 be a density with support on $[0, \infty)$, and let $F_0(x) = \int_0^x f_0(u)du$. Then \hat{f}_0 exists and is unique. Moreover, \hat{f}_0 is found as the (left) derivative of \hat{F}_0 , the least concave majorant of F_0 .*

Thus, in our setting, we have a complete graphical representation of the distribution function \hat{F}_0 of the KL projection. This representation makes it possible to calculate \hat{f}_0 in many cases. It also allows us to easily visualize the various F_0 which yield the same \hat{f}_0 . Moreover, the representation is key in understanding the dynamics of the estimator, both on a finite sample and asymptotic level. Therefore, for a function g we define the operator $\text{gren}(g)$ to denote the (left) derivative of the least concave majorant of g . When the least concave majorant is restricted to a set $[a, b]$, we will write $\text{gren}_{[a,b]}(g)$.

Let \mathcal{S}_0 denote the support of f_0 . We write $\mathcal{S}_0 = \mathcal{M} \cup \mathcal{W}$, where $\mathcal{M} = \{x \geq 0 : \hat{F}_0(x) > F_0(x)\}$ and $\mathcal{W} = \{x \geq 0 : \hat{F}_0(x) = F_0(x)\}$. Since f_0 is a density, it follows that F_0 is continuous, as is \hat{F}_0 , and therefore \mathcal{W} is a closed set and \mathcal{M} is open. For a fixed point $x_0 \in \mathcal{M}$, we thus know that x_0 lies in some open interval. Indeed, let $a_0 = \sup\{x < x_0 : \hat{F}_0(x_0) = F_0(x_0)\}$ and $b_0 = \inf\{x > x_0 : \hat{F}_0(x_0) = F_0(x_0)\}$. Then $x_0 \in (a_0, b_0)$ with $(a_0, b_0) \subset \mathcal{M}$.

Two examples are given in Figure 1. For the first example we have

$$f_0(x) = \begin{cases} 1.5 & x \in [0, 0.5] \\ x - 0.25 & x \in (0.5, 1]. \end{cases} \quad (2.1)$$

Here $\mathcal{M} = (0.5, 1)$ and $\mathcal{W} = [0, 0.5] \cup \{1\}$.

$$f_0(x) = \begin{cases} 12(x - 0.5)^2 & x \in [0, 0.4] \cup [0.6, 1] \\ 0.04 & x \in (0.4, 0.6). \end{cases} \quad (2.2)$$

Here $\mathcal{M} = (0.25, 1)$ and $\mathcal{W} = [0, 0.25] \cup \{1\}$. The next proposition gives some additional properties of the KL projection.

Proposition 2.2. *The KL projection density, \hat{f}_0 , satisfies the following:*

- (1) *Fix $x_0 \in \mathcal{M}$ and define a_0, b_0 as above. Then $b_0 < \infty$, and \hat{f}_0 is constant on $(a_0, b_0]$ and satisfies the mean-value property*

$$\hat{f}_0(x_0) = \frac{1}{b_0 - a_0} \int_{a_0}^{b_0} f_0(x) dx.$$

- (2) *Suppose that $\int_0^\infty f_0^2(x) dx < \infty$. Then $\hat{f}_0 = \operatorname{argmin}_{g \in \mathcal{F}} \int_0^\infty (g(x) - f_0(x))^2 dx$.*
 (3) *For any increasing function $h(x)$, $\int_0^\infty h(x) \hat{f}_0(x) dx \leq \int_0^\infty h(x) f_0(x) dx$.*
 (4) *Let Φ denote a non-negative convex function and suppose that g_0 is decreasing. Then*

$$\int \Phi(\hat{f}_0(x) - g_0(x)) dx \leq \int \Phi(f_0(x) - g_0(x)) dx.$$

- (5) *Let $g_0 \in \mathcal{F}$ and let $G_0(y) = \int_0^y g_0(x) dx$. Then*

$$\sup_{x \geq 0} |\hat{F}_0(x) - G_0(x)| \leq \sup_{x \geq 0} |F_0(x) - G_0(x)|.$$

Point (3) above tells us that if g is increasing then $\mu_0(g) \geq \hat{\mu}_0(g)$. Point (5) is Marshall's Lemma (Marshall, 1970). The proof of Proposition 2.2 appears in the Appendix. The next theorem is our first main result.

Theorem 2.3. *Fix a point $x_0 \in \mathcal{M}$, and let $[a, b]$ denote the largest interval I such that $\hat{F}_0(x)$ is linear on I . Let \mathbb{U} denote a standard Brownian bridge process on $[0, 1]$, and let $\mathbb{U}_{F_0}(x) = \mathbb{U}(F_0(x))$ for $x \in \mathcal{S}$. Then*

$$\sqrt{n}(\hat{f}_n(x_0) - \hat{f}_0(x_0)) \Rightarrow \operatorname{gren}_{[a,b]}(\mathbb{U}_{F_0}^{mod})(x_0),$$

where

$$\mathbb{U}_{F_0}^{mod}(u) = \begin{cases} \mathbb{U}_{F_0}(u) & u \in [a, b] \cap \mathcal{W}, \\ -\infty & u \in [a, b] \cap \mathcal{M}. \end{cases}$$

If it happens that $[a, b] \cap \mathcal{W} = \{a, b\}$, then

$$\sqrt{n}(\hat{f}_n(x_0) - \hat{f}_0(x_0)) \Rightarrow \sigma Z,$$

where Z is a standard normal random variable and

$$\sigma^2 = \hat{f}_0(x_0) \left[\frac{1}{b-a} - \hat{f}_0(x_0) \right].$$

Recall that Patilea (2001, Corollary 5.6) shows that the rate of convergence (in Hellinger distance) of \hat{f}_n to \hat{f}_0 is $n^{-1/3}$. The above theorem shows that the *local* rate of convergence will be \sqrt{n} where the KL projection is flat. When the KL density is curved, the KL density and true density are actually equal, and hence the convergence rate from the correctly specified case applies. The next formulation of the limiting process is similar to that of Carolan and Dykstra (1999) for a density with a flat region on $[a, b]$.

Remark 2.4. Let $\hat{p}_0 = \hat{F}_0(b) - \hat{F}_0(a) = F_0(b) - F_0(a)$. Since \hat{F}_0 is linear on $[a, b]$ the limiting distribution may also be expressed as

$$\text{gren}_{[a,b]}(\mathbb{U}_{F_0}^{\text{mod}})(x_0) = \frac{1}{b-a} \left\{ Z + \sqrt{\hat{p}_0} \text{gren}(\mathbb{U}^{\text{mod}}) \left(\frac{x_0 - a}{b-a} \right) \right\},$$

where Z is a mean zero normal random variable with variance $\hat{p}_0(1 - \hat{p}_0)$, \mathbb{U} is an independent standard Brownian bridge, and

$$\mathbb{U}^{\text{mod}}(u) = \begin{cases} \mathbb{U}(u) & u \in ([a, b] \cap \mathcal{W} - a)/(b-a), \\ -\infty & u \in ([a, b] \cap \mathcal{M} - a)/(b-a). \end{cases}$$

Notably, if $[a, b] \cap \mathcal{W} = \{a, b\}$, then $\text{gren}(\mathbb{U}^{\text{mod}})(u) = 0$.

Figure 2 illustrates the theory. The convergence is surprisingly fast, although it appears to be a little slower in the second example (2.2). We postulate that this difference is caused by the presence/absence of the strictly curved region of f_0 .

Proof of Theorem 2.3. By the switching relation (Balabdaoui et al., 2011), we have

$$\begin{aligned} & P \left(\sqrt{n}(\hat{f}_n(x_0) - \hat{f}_0(x_0)) < t \right) \\ &= P \left(\operatorname{argmax}_{z \geq 0} \left\{ \mathbb{F}_n(z) - (\hat{f}_0(x_0) + n^{-1/2}t)z \right\} < x \right) \\ &= P \left(\operatorname{argmax}_{z \geq 0} \left\{ \sqrt{n}(\mathbb{F}_n(z) - \mathbb{F}_n(a) - (F_0(z) - F_0(a))) \right. \right. \\ &\quad \left. \left. + \sqrt{n} \left(F_0(z) - F_0(a) - \hat{f}_0(x_0)(z-a) \right) - tz \right\} < x \right). \end{aligned}$$

We now look more closely at the “second” term. That is,

$$\begin{aligned} & F_0(z) - F_0(a) - \hat{f}_0(x_0)(z-a) \\ &= - \left\{ \hat{F}_0(z) - F_0(z) \right\} + \left\{ \hat{F}_0(z) - \hat{F}_0(a) - \hat{f}_0(x_0)(z-a) \right\}, \end{aligned}$$

noting that $\hat{F}_0(a) = F_0(a)$, since $a \in [a, b] \cap \mathcal{W}$. On the other hand, for all $z \in [a, b] \cap \mathcal{M}$, we have $\hat{F}_0(z) > F_0(z)$. Furthermore, \hat{F}_0 is concave with derivative $\hat{f}_0(x_0)$ (at any point

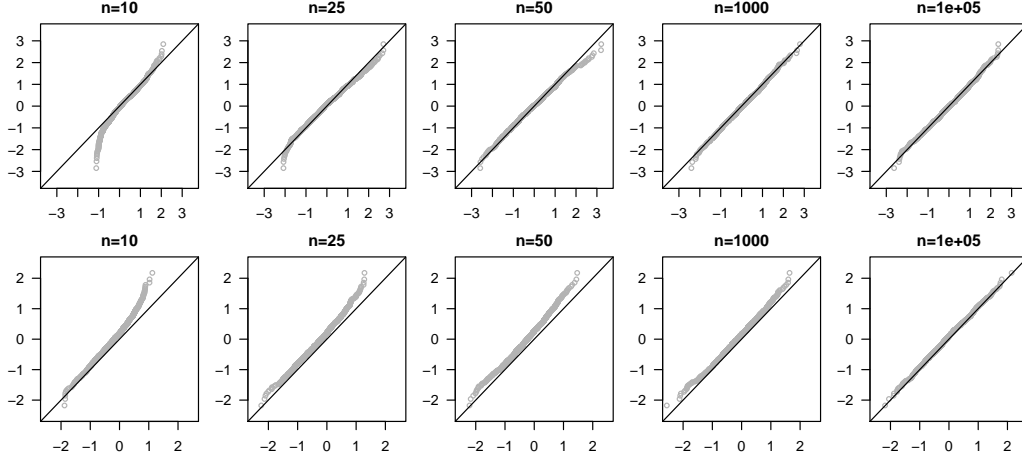


FIGURE 2. Empirical quantiles of $\sqrt{n}(\hat{f}_n(x_0) - \hat{f}_0(x_0))$ vs. the true quantiles of the limiting distribution at the point $x_0 = 0.75$ for f_0 given by (2.1) in the top row and (2.2) in the bottom row. The sample size varies from $n = 10$ to $n = 100\,000$. The straight line goes through the origin and has slope one. Each plot is based on $B = 1000$ samples.

$z \in (a, b)$, and hence

$$\hat{F}_0(z) - \hat{F}_0(a) - \hat{f}_0(x_0)(z - a) \leq 0$$

for all z . For $z \in [a, b] \cap \mathcal{W}$ this is an equality, and a strict inequality otherwise. Therefore, the weak limit of

$$\sqrt{n} \{(\mathbb{F}_n(z) - \mathbb{F}_n(a) - (F_0(z) - F_0(a)))\} - \sqrt{n} (F_0(z) - F_0(a) - \hat{f}_0(x_0)(z - a))$$

is

$$\mathbb{U}_{F_0}^{mod}(z) - \mathbb{U}_{F_0}^{mod}(a) = \mathbb{U}_{F_0}^{mod}(z) - \mathbb{U}_{F_0}(a),$$

for all $z \in [a, b]$. For $z \notin [a, b] \cap \mathcal{W}$, the limit of this process is always $-\infty$, and therefore the maximum must occur inside of $[a, b]$. Thus,

$$\begin{aligned} P\left(\sqrt{n}(\hat{f}_n(x_0) - \hat{f}_0(x_0)) < t\right) &\rightarrow P\left(\operatorname{argmax}_{z \in [a, b]} \{\mathbb{U}_{F_0}^{mod}(z) - tz\} < x\right) \\ &= P\left(\operatorname{gren}_{[a, b]}(\mathbb{U}_{F_0}^{mod})(x_0) < t\right), \end{aligned}$$

by switching again. When $[a, b] \cap \mathcal{W} = \{a, b\}$, then the least concave majorant is simply the line joining $\mathbb{U}_{F_0}(a)$ and $\mathbb{U}_{F_0}(b)$, with slope equal to

$$\frac{\mathbb{U}_{F_0}(b) - \mathbb{U}_{F_0}(a)}{b - a},$$

a Gaussian random variable with mean zero and variance

$$\begin{aligned} & \frac{1}{(b-a)^2} (F_0(b) - F_0(a)) [1 - (F_0(b) - F_0(a))] \\ &= \frac{1}{(b-a)^2} (\widehat{F}_0(b) - \widehat{F}_0(a)) [1 - (\widehat{F}_0(b) - \widehat{F}_0(a))] = \widehat{f}_0(x_0) \left[\frac{1}{b-a} - \widehat{f}_0(x_0) \right]. \end{aligned}$$

□

Proof of Remark 2.4. Recall that \widehat{F}_0 is linear on $[a, b]$. Therefore, for $x \in [a, b]$, we can write $\mathbb{U}(\widehat{F}_0(x)) - \mathbb{U}(\widehat{F}_0(a)) = \frac{x-a}{b-a} \mathbb{W} + \mathbb{V}(x)$, where

$$\begin{aligned} \mathbb{W} &= \mathbb{U}(\widehat{F}_0(b)) - \mathbb{U}(\widehat{F}_0(a)), \\ \mathbb{V}(x) &= \mathbb{U}(\widehat{F}_0(x)) - \mathbb{U}(\widehat{F}_0(a)) - \frac{\widehat{F}_0(x) - \widehat{F}_0(a)}{\widehat{F}_0(b) - \widehat{F}_0(a)} \mathbb{W} \\ &= \mathbb{U}(\widehat{F}_0(x)) - \mathbb{U}(\widehat{F}_0(a)) - \frac{x-a}{b-a} \mathbb{W}. \end{aligned}$$

Since all variables are jointly Gaussian, a careful calculation of the covariances reveals that \mathbb{W} and $\mathbb{V}(x)$ are independent (also as processes), and \mathbb{W} is mean-zero Gaussian with variance $\widehat{p}_0(1 - \widehat{p}_0)$. Furthermore,

$$\mathbb{V}(s) \stackrel{d}{=} \sqrt{\widehat{p}_0} \mathbb{U} \left(\frac{s-a}{b-a} \right).$$

This decomposition is similar to that of Shorack and Wellner (1986, Exercise 2.2.11, page 32). Now, note that the Grenander operator satisfies $\text{gren}_{[a,b]}(g)(x) = \beta + \frac{\gamma}{b-a} \text{gren}_{[0,1]}(h) \left(\frac{t-a}{b-a} \right)$ if $g(t) = \alpha + \beta t + \gamma h \left(\frac{t-a}{b-a} \right)$. It follows that

$$\text{gren}_{[a,b]}(\mathbb{U}_{\widehat{F}_0})(x_0) = \frac{1}{b-a} Z + \frac{\sqrt{\widehat{p}_0}}{b-a} \text{gren}(\mathbb{U}) \left(\frac{x_0 - a}{b-a} \right),$$

with Z, \mathbb{U} defined as in the Remark. The full result follows since, $\mathbb{U}_{F_0}^{mod}(x) = \mathbb{U}_{\widehat{F}_0}^{mod}(x) = \frac{x-a}{b-a} \mathbb{W} + \mathbb{V}^{mod}(x)$. □

3. $n^{-1/2}$ CONVERGENCE OF LINEAR FUNCTIONALS

Consider a density f_0 with support \mathcal{S}_0 and let \widehat{f}_0 denote its KL projection. The results for linear functionals hold under the following assumptions.

(S1). The support, \mathcal{S}_0 , of f_0 is bounded.

We write $\mathcal{S}_0 = \mathcal{S}_c \cup \mathcal{S}_f$, where \mathcal{S}_c denotes the portion of the support where \widehat{f}_0 is curved and \mathcal{S}_f denotes the portion of the support where \widehat{f}_0 is flat. We assume also that f_0 is differentiable on \mathcal{S}_c .

(C1). Define $\gamma_0 = \sup_{x \in \mathcal{S}_c} |f_0'(x)| / \inf_{x \in \mathcal{S}_c} f_0^2(x)$. Then $\gamma_0 < \infty$.

(C2). Define $\beta_0 = \inf_{x \in \mathcal{S}_c} |f_0'(x) / f_0^2(x)|$. Then $\beta_0 > 0$.

(F1). On \mathcal{S}_f , the KL projection can be written as

$$\hat{f}_0(x) = \sum_{j=1}^J \hat{q}_j 1_{I_j}(x)$$

where $\hat{q}_1 > \hat{q}_2 > \dots > \hat{q}_J > 0$, J is finite, and the intervals are disjoint and each is of the form $I_j = (a_j, b_j]$.

(F2) $f_0(x) > 0$ for all $x \in \mathcal{S}_f \cap \mathcal{M}$.

Let $g : \mathcal{S}_0 \mapsto \mathbb{R}$ and define $\hat{\mu}_n(g)$ by (1.1). Then we require that g satisfy the following conditions.

(G1). g is differentiable on \mathcal{S}_c with $\int_{\mathcal{S}_c} |g'(x)| dx < \infty$ and $\sup_{\mathcal{S}_c} |g(x)| < \infty$.

(G2). $g \in L_\beta(\mathcal{S}_f)$ for some $\beta > 2$.

In order to state our main result for linear functionals we need to define the following functions,

$$\begin{aligned} g_j(u) &= g((b_j - a_j)u + a_j) \quad u \in [0, 1], \\ \bar{g}_j &= (b_j - a_j)^{-1} \int_{a_j}^{b_j} g(x) dx. \end{aligned} \tag{3.1}$$

$$\bar{g}(x) = \begin{cases} g(x) & x \in \mathcal{S}_c, \\ \bar{g}_j & x \in I_j, \quad j = 1, \dots, J. \end{cases} \tag{3.2}$$

Thus, $\bar{g}_1, \dots, \bar{g}_J$ are the local averages of the function g , and each $g_j(u)$ is a localized version of g .

Theorem 3.1. *Suppose that the density f_0 satisfies conditions (S1), (C1), (C2), (F1), and (F2). Consider a function $g : \mathcal{S}_0 \mapsto \mathbb{R}$ which satisfies conditions (G1) and (G2). Let $\mathbb{U}, \mathbb{U}_1, \dots, \mathbb{U}_J$ denote independent Brownian bridges, $\mathbb{U}_{F_0}(x) = \mathbb{U}(F_0(x))$, and define \mathbb{U}_j^{mod} as in Theorem 2.3. Then*

$$\sqrt{n}(\hat{\mu}_n(g) - \hat{\mu}_0(g)) \Rightarrow \int_{\mathcal{S}_0} \bar{g}(x) d\mathbb{U}_{F_0}(x) + \sum_{j=1}^J \sqrt{p_j} \int_0^1 g_j(u) \text{gren}(\mathbb{U}_j^{mod})(u) du.$$

Furthermore, $\int_{\mathcal{S}_0} \bar{g}(x) d\mathbb{U}_{F_0}(x) = \int_{\mathcal{S}_0} \bar{g}(x) d\mathbb{U}_{\hat{F}_0}(x)$. Also, if $I_j \cap \mathcal{W} = \{a_j, b_j\}$, then $\text{gren}(\mathbb{U}_j^{mod}) \equiv 0$.

It follows that $\sqrt{n}(\hat{\mu}_n(g) - \hat{\mu}_0(g))$ will converge to a Gaussian limit for true density (2.2) but not for (2.1), as the latter has well-specified flat regions. A simulation for (2.2) is shown in Figure 3. The proof of Theorem 3.1 is given in Section 5. The simulations show that there appears a systematic bias prior to convergence (the empirical quantiles appear on the x -axis in Figure 3, the negative bias translates to a left-shift in the plot). The proof of Proposition 5.1 shows that one source of the bias is the term $\sqrt{n} \int x d(\hat{F}_n - \mathbb{F}_n) \approx -\sqrt{n} \int (\hat{F}_n - \mathbb{F}_n) \leq 0$. When $\mathcal{S}_0 = \mathcal{S}_c$, this term is the only source of bias, and from Kiefer and Wolfowitz (1976), it converges to zero at a

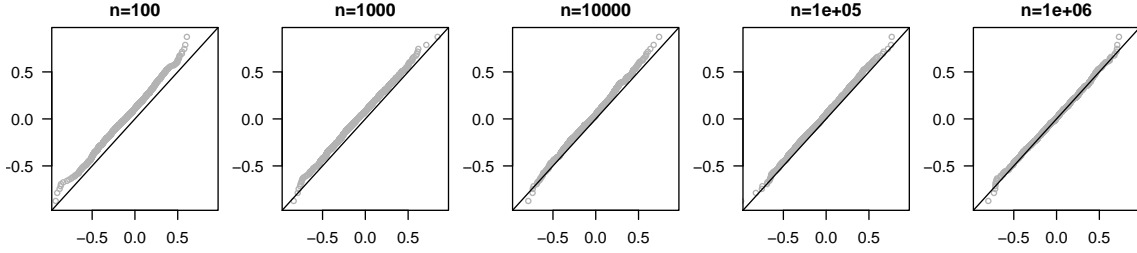


FIGURE 3. Empirical quantiles of $\sqrt{n}(\hat{\mu}_n(g) - \hat{\mu}_0(g))$ with $g(x) = x$ vs. the true quantiles of the limiting distribution for f_0 given in (2.2).

rate of at least $n^{-1/6}(\log n)^{2/3}$. Since (3) of Proposition 2.2 also holds at the empirical level, similar behavior will be seen for all increasing functions g .

Finally, we make a few comments on the assumptions required for Theorem 3.1 to hold. These do appear long, but have been split up to facilitate reading of the different assumptions which are required on the different portions of the KL density \hat{f}_0 . The assumption that \mathcal{S}_0 is bounded, along with (C1) and (C2) are standard in order for results to hold when f_0 is “curved”. The particular form is taken from Kiefer and Wolfowitz (1976), but some version of these has been assumed in either a global or local sense when proving asymptotics by Durot et al. (2012); Kulikov and Lopuhaä (2008), or Groeneboom et al. (1999), to name but a few. It may be possible to relax these, perhaps using the ideas in Kiefer and Wolfowitz (1977), but this is beyond the scope of the current work. The strongest part of assumption (F1) is that J is finite, while (F2) is required to get the tight exponential bounds under misspecification. Assumptions (G1) and (G2) are what is required of the function g in the linear functional, and these are different on the curved and flat regions. Since \mathcal{S}_0 is bounded, numerous functions will satisfy both (G1) and (G2).

We also note that in the case that $\mathcal{S}_0 = \mathcal{S}_c$, the ideas of van de Geer (2003) are of interest. van de Geer (2003) gives an overview of when one can expect asymptotic efficiency of plug-in estimators, such as $\hat{\mu}_n(g)$. Although the requirements for asymptotic efficiency do depend on g , the main requirement of f_0 is that it have no “flat” regions. The cases considered in van de Geer (2003) are of the well-specified kind, but we now know that this will always be the case when the KL projection density is curved.

Remark 3.2. *Marginal properties of the process $\text{gren}(\mathbb{U})$ were studied in Carolan and Dykstra (2001). The results include marginal densities and moments, and in particular $E[\text{gren}(\mathbb{U})(x)] = 4(1-2x)/3\sqrt{2\pi x(1-x)}$ and $E[(\text{gren}(\mathbb{U})(x))^2] = 0.5(x^2/(1-x) + (1-x)^2/x)$. Note that it follows that $E[\int_0^1 (\text{gren}(\mathbb{U})(x))^2 dx] = \int_0^1 (1-x)^2/x dx = \infty$, and hence the process $\text{gren}(\mathbb{U})$ is not defined in L_2 . We would therefore not expect convergence of $\hat{\mu}_n(g)$ for $g \in L_\beta(\mathcal{S}_f)$ with $\beta \in [1, 2]$.*

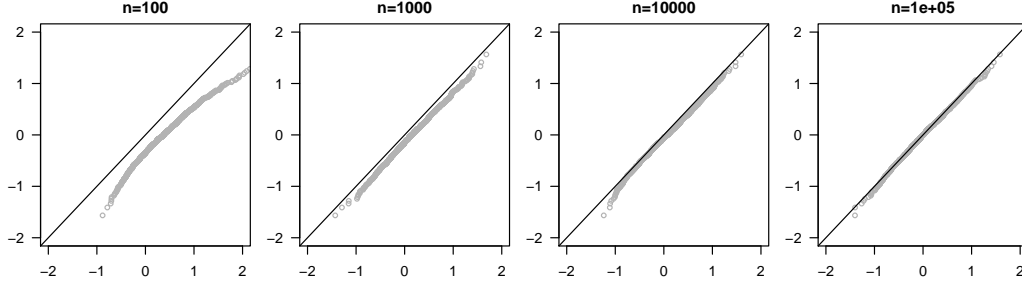


FIGURE 4. Empirical quantiles of $\sqrt{n}(T(\hat{f}_n) - T(\hat{f}_0))$ vs. the true quantiles of the limiting distribution for f_0 given in (2.1).

4. BEYOND LINEAR FUNCTIONALS: A SPECIAL CASE

Entropy measures the amount of disorder or uncertainty in a system and is closely related to the Kullback-Liebler divergence. Let $T(f) = \int_0^\infty f(x) \log f(x) dx$ denote the entropy functional. A modified version of the entropy functional was used in Cule et al. (2010) in the context of testing for log-concavity. A review of testing and other applications of entropy appears for example in Beirlant et al. (1997).

Theorem 4.1. *Suppose that \hat{f}_0 is bounded, the support of f_0 is also bounded, and that $f_0/\hat{f}_0 \leq c_0^2 < \infty$. Then*

$$\sqrt{n} \left(T(\hat{f}_n) - T(\hat{f}_0) \right) \Rightarrow \sigma Z,$$

where Z is a standard normal random variable and

$$\sigma^2 = \text{Var}_{f_0}(\log(\hat{f}_0(X))) = \text{Var}_{\hat{f}_0}(\log(\hat{f}_0(X))).$$

The proof is made up of two key pieces: (1) tight bounds on the likelihood ratio from Lemma 4.2 and (2) specialized equalities which hold for the Grenander estimator.

Lemma 4.2. *Suppose that \hat{f}_0 is bounded, the support of f_0 is also bounded, and that $f_0/\hat{f}_0 \leq c_0^2 < \infty$. Then*

$$\int \log \frac{\hat{f}_n}{\hat{f}_0} d\mathbb{F}_n = O_p(n^{-2/3}).$$

We note that the conditions we require here are stronger than those of Patilea (2001, Corollary 5.6). However, under those conditions Patilea (2001) establishes convergence rates on $\int \log \frac{2\hat{f}_n}{\hat{f}_n + \hat{f}_0} d\mathbb{F}_n$, which is not sufficient for our purposes. The condition that f_0/\hat{f}_0 is bounded above was also used in the study of misspecification in van de Geer (2000, Section 10.4). The condition that the support of f_0 is bounded is the strongest, whereas the condition that \hat{f}_0 is bounded may be relaxed somewhat. We discuss this further at the end of Section 6.2.1 in the Appendix.

Proof. We first show that $\int_0^\infty \varphi(\hat{f}_n) d(\hat{F}_n - \mathbb{F}_n) = 0$ for any function φ . This follows since $\hat{F}_n(x) \geq \mathbb{F}_n(x)$ with equality at finitely many touch points, and also \hat{f}_n is constant between all touch points. Thus, letting τ_i enumerate the (random) points of touch, we have

$$\int_0^\infty \varphi(\hat{f}_n) d(\hat{F}_n - \mathbb{F}_n) = \sum_{i=1}^m \varphi(\hat{f}_n(\tau_i)) \left((\hat{F}_n - \mathbb{F}_n)(\tau_i) - (\hat{F}_n - \mathbb{F}_n)(\tau_{i-1}) \right) = 0,$$

with $\tau_0 = 0$ and $\tau_m = X_{(n)}$. A similar argument also establishes that

$$\int_0^\infty \varphi(\hat{f}_0) d(\hat{F}_0 - F_0) = \int_{\mathcal{M}} \varphi(\hat{f}_0) d(\hat{F}_0 - F_0) = 0. \quad (4.1)$$

For $\varphi(v) = \log v$, it follows that

$$\begin{aligned} \sqrt{n} \left(T(\hat{f}_n) - T(\hat{f}_0) \right) &= \sqrt{n} \left(\int \log \hat{f}_n d\hat{F}_n - \int \log \hat{f}_0 d\hat{F}_0 \right) \\ &= \sqrt{n} \int \log \left(\frac{\hat{f}_n}{\hat{f}_0} \right) d\mathbb{F}_n + \sqrt{n} \int \log \hat{f}_0 d(\mathbb{F}_n - F_0). \end{aligned}$$

The first term is $O_p(n^{-1/6})$ by Lemma 4.2. The second term has a Gaussian limit with variance $\text{Var}_{f_0}(\log \hat{f}_0(X))$. By (4.1) (with $\varphi(v) = \log^2 v, \log v$) this is equal to $\text{Var}_{\hat{f}_0}(\log \hat{f}_0(X))$. \square

A simulation of this result is shown in Figure 4 based on the true density (2.1). Note that this density has well-specified flat regions, and therefore linear functionals that do not ignore $\mathcal{S}_f \cap \mathcal{W}$ should have non-Gaussian terms in their limit. On the other hand, the entropy functional will *always* result in a Gaussian limit. The simulations exhibit a systematic positive bias. The proof shown above reveals the cause: The term $\int \log(\hat{f}_n/\hat{f}_0) d\mathbb{F}_n \geq 0$ since \hat{f}_n is the MLE. In the plots the quantiles of $\sqrt{n}(T(\hat{f}_n) - T(\hat{f}_0))$ are shown on the x -axis, and these quantiles appear to be shifted to the right – that is, they are larger than the quantiles of the limiting Gaussian distribution.

5. PROOFS FOR SECTION 3

We now present the proof for Theorem 3.1. We proceed by proving convergence results for the different types of behaviors of the density separately (curved, flat, misspecified), and combine the results together at the end. We believe that the intermediate results are of independent interest to the reader, and we also hope that this approach makes the proof more accessible.

5.1. Strictly curved well-specified density. We first suppose that the true density f_0 satisfies the conditions introduced in Kiefer and Wolfowitz (1976).

Proposition 5.1. *Suppose that the support $\mathcal{S}_0 = \mathcal{S}_c$ is bounded and that f_0 satisfies conditions (C1) and (C2). Suppose further that g satisfies condition (G1). Then*

$$\sqrt{n}(\hat{\mu}_n(g) - \mu_0(g)) \Rightarrow \sigma Z,$$

where Z is a standard normal random variable and $\sigma^2 = \text{Var}(g(X)) < \infty$.

We note that this result is similar to that in Kulikov and Lopuhaä (2008). The conditions required in Kulikov and Lopuhaä (2008) are slightly different than those used in Kiefer and Wolfowitz (1976).

Proof. Without loss of generality, we assume that $\mathcal{S}_0 = \mathcal{S}_c = [0, 1]$. Let $\bar{\mu}_n(g) = n^{-1} \sum_{i=1}^n g(X_i)$ denote the empirical estimator of $\mu_0(g)$. Using integration by parts, we have

$$\begin{aligned} |\hat{\mu}_n(g) - \bar{\mu}_n(g)| &= \left| g(x)[\hat{F}_n(x) - \mathbb{F}_n(x)] \Big|_{x=0}^{x=1} + \int_0^1 g'(x) [\hat{F}_n(x) - \mathbb{F}_n(x)] dx \right| \\ &\leq \left\{ \max_{x \in \partial \mathcal{S}_c} |g(x)| + \int_{\mathcal{S}_c} |g'(x)| dx \right\} \sup_{x \in \mathcal{S}_c} |\hat{F}_n(x) - \mathbb{F}_n(x)|. \end{aligned}$$

Now, by the results of Kiefer and Wolfowitz (1976), we have $\sup_{x \in [0, 1]} \sqrt{n}|\hat{F}_n(x) - \mathbb{F}_n(x)| = o_p(n^{-1/6} \log^{2/3} n)$. Therefore,

$$\sqrt{n}(\hat{\mu}_n(g) - \mu_0(g)) = \sqrt{n}(\bar{\mu}_n(g) - \mu_0(g)) + o_p(n^{-1/6} \log^{2/3} n),$$

from which the result follows. \square

5.2. Piecewise constant well-specified density. Suppose next that $\mathcal{S}_0 = \mathcal{S}_f = \mathcal{W} \cap \mathcal{S}_0$. That is, the true density is piecewise constant decreasing and can be expressed as

$$f_0(x) = \sum_{j=1}^J q_j 1_{(a_j, b_j]}(x) \tag{5.1}$$

where $q_1 > q_2 > \dots > q_J > 0$ and $\cup I_j = \mathcal{S}_0$ where the sets $I_j = (a_j, b_j]$ are disjoint. Indeed, we have $a_1 = 0$ and $a_{j+1} = b_j$. Notice that $1 = \sum_{j=1}^J q_j(b_j - a_j)$. We also define

$$p_j = q_j(b_j - a_j) = P(a_j < X \leq b_j),$$

where X is a random variable with density f_0 . Also, let $\mathbb{U}_1, \dots, \mathbb{U}_J$ denote independent standard Brownian bridge processes (each defined on $[0, 1]$), and let $\{Z_1, \dots, Z_J\}$ be an independent multivariate normal with mean zero and covariance $\text{diag}(p) - pp^T$ for $p = (p_1, \dots, p_J)^T$.

Proposition 5.2. *Suppose that f_0 is as in (5.1). Then $\sqrt{n}(\hat{f}_n(x) - f_0(x))$ converges weakly to $\mathbb{S}(x)$ in $L_\alpha(\mathcal{S}_f) = L_\alpha(\mathcal{S}_0)$ for any $\alpha \in [1, 2)$, where*

$$\mathbb{S}(x) = \sum_{j=1}^J \frac{1}{b_j - a_j} \left\{ Z_j + \sqrt{p_j} \text{gren}(\mathbb{U}_j) \left(\frac{x - a_j}{b_j - a_j} \right) \right\} 1_{I_j}(x).$$

A pointwise version of Proposition 5.2 was originally proved in Carolan and Dykstra (1999). Here, we extend these results to convergence in L_α , which is a much stronger statement, requiring tight bounds on the tail behavior at a point of the kind proved in Groeneboom et al. (1999, Theorem 2.1). In the case of the decreasing probability mass function, ℓ_k , $k \geq 1$ convergence has been established in Jankowski and Wellner (2009). An immediate corollary of this work is convergence of the linear functionals $\hat{\mu}_n(g)$: see Corollary 5.3 below.

Groeneboom (1986, Theorem 4.1) shows that for f_0 equal to the uniform density on $[0, 1]$ we have

$$\sqrt{n} \int_0^1 |\hat{f}_n(x) - f_0(x)| dx \Rightarrow \int_0^1 |\text{gren}(\mathbb{U})(x)| dx = 2 \sup_{0 \leq x \leq 1} \mathbb{U}(x),$$

where \mathbb{U} is again a standard Brownian bridge process on $[0, 1]$. This is an immediate corollary of Proposition 5.2 with $J = 1$. On the other hand, Groeneboom (1983) (see also Groeneboom and Pyke (1983)) shows that

$$\frac{\int_0^1 (\sqrt{n}(\hat{f}_n(x) - f_0(x)))^2 dx - \log n}{\sqrt{3 \log n}} \Rightarrow Z \sim N(0, 1)$$

and hence convergence of $\sqrt{n}(\hat{f}_n(x) - f_0(x))$ to $\text{gren}(\mathbb{U})(x)$ in $L_2([0, 1])$ fails. See also Remark 3.2.

Corollary 5.3. *Suppose that f_0 takes the form (5.1) with bounded support $\mathcal{S}_0 = \mathcal{S}_f \cap \mathcal{W}$ and with a finite J . Suppose further that g satisfies condition (G2). Then $\sqrt{n}(\hat{\mu}_n(g) - \mu_0(g)) \Rightarrow Y_J$, where*

$$Y_J = \sum_{j=1}^J \left\{ \bar{g}_j Z_j + \sqrt{p_j} \int_0^1 g_j(u) \text{gren}(\mathbb{U}_j)(u) du \right\},$$

with \bar{g}_j and g_j defined in (3.1).

In the proofs which follow, unless stated otherwise, we assume that $\mathcal{S}_0 = [0, 1]$.

Lemma 5.4. *Suppose that f_0 is as in (5.1) with a discontinuity at a point $x_0 \neq 0$. Then, for all $c > 0$,*

$$\sup_{0 \leq x \leq c/n} \left| \hat{f}_n(x_0 + x) - f_0(x_0 + x) \right| = O_p(1).$$

Proof. It was shown in Anevski and Hössjer (2002, Theorem 2) that

$$\widehat{f}_n(x_0 + t/n) - \frac{f_0(x_0-) + f_0(x_0+)}{2} \Rightarrow h(t), \quad (5.2)$$

where $h(t)$ is the left derivative of the least concave majorant (over \mathbb{R}) of the process

$$\mathbb{N}(\lambda(s)) - \lambda(s) - \left\{ \frac{f_0(x_0-) - f_0(x_0+)}{2} \right\} |s|,$$

where the rate function is equal to

$$\lambda(s) = \begin{cases} f_0(x_0+)s, & s > 0, \\ f_0(x_0-)s, & s < 0. \end{cases}$$

Here, \mathbb{N} denotes a standard two-sided Poisson process. The result in Anevski and Hössjer (2002, Theorem 2) is established by a “switching” argument similar to that in the proof of Theorem 2.3. The switching argument can also be extended to this situation even if $f_0(x_0-) = f_0(x_0+)$. A similar argument may also be used to show convergence in finite dimensional distributions as well. We next show convergence of the supremum norm

$$\sup_{0 \leq x \leq c/n} \left| \widehat{f}_n(x_0 + x) - f_0(x_0 + x) \right| \Rightarrow \sup_{0 \leq x \leq c} \left| h(x) + \frac{f_0(x_0-) - f_0(x_0+)}{2} \right| = O_p(1).$$

This is done by (1) showing that the convergence in (5.2) also holds in $D[0, \infty)$, and (2) showing that this implies convergence of the supremum norm (as above). Both of these steps follow exactly the same argument as the proof of Theorem 1.1 in Balabdaoui et al. (2011), and we therefore omit the details. \square

Lemma 5.5. *Suppose that f_0 is decreasing on \mathbb{R} and flat on $(a, b]$ and fix $x \in (a, b)$. Then, for any $t_0 > 0$ and $k_0 > 0$, there exists a constant $c_0 = t_0/(f_0(b) + t_0/k_0)$ such that*

$$P\left(\widehat{f}_n(x) > f_0(x) + n^{-1/2}t\right) \leq \exp\left\{-c_0 \frac{t(x - x_0)}{2}\right\} \quad \text{for all } t \geq t_0,$$

for all $n \geq (k_0/3)^2$. Also,

$$P\left(\widehat{f}_n(x) < f_0(x) - n^{-1/2}t\right) \leq \exp\left\{-\frac{t^2(b - x)}{2f_0(x)}\right\} \quad \text{for all } t \in [0, \sqrt{n}f_0(x)],$$

and otherwise the probability is equal to zero.

Proof. Let $\mathbb{F}_n(a, s) = \mathbb{F}_n(s) - \mathbb{F}_n(a)$, and we write $f_0(a+) = \lim_{x \rightarrow a+} f_0(x)$. By the switching relation,

$$\begin{aligned}
P\left(\widehat{f}_n(x) > f_0(x) + n^{-1/2}t\right) &= P\left(\operatorname{argmax}_{s \in [0,1]} \{\mathbb{F}_n(s) - (f_0(x) + n^{-1/2}t)s\} > x\right) \\
&= P\left(\operatorname{argmax}_{s \in [0,1]} \{\mathbb{F}_n(a, s) - (f_0(a+) + n^{-1/2}t)(s - a)\} > x\right) \\
&\leq P\left(\mathbb{F}_n(a, s) \geq (f_0(a+) + n^{-1/2}t)(s - a) \text{ for some } s \in (x, 1]\right) \\
&= P\left(\frac{\mathbb{F}_n(a, s)}{F_0(a, s)} \geq \frac{(f_0(a+) + n^{-1/2}t)(s - a)}{F_0(a, s)} \text{ for some } s \in (x, 1]\right) \\
&= P\left(\frac{\mathbb{F}_n(a, s)}{F_0(a, s)} \geq 1 + \frac{n^{-1/2}t}{f_0(a+)} \text{ for some } s \in (x, 1]\right) \\
&\leq P\left(\sup_{s \in (x, 1]} \frac{\mathbb{F}_n(a, s)}{F_0(a, s)} \geq 1 + \frac{n^{-1/2}t}{f_0(a+)}\right)
\end{aligned}$$

Since $\mathbb{F}_n(a, s)$ is a binomial random variable, we can bound the above using Shorack and Wellner (1986, Inequality 10.3.2, page 416), with $h(v) = v(\log v - 1) + 1$ and $\psi(v) = 2h(1+v)/v^2 \geq (1+v/3)^{-1}$. It therefore follows that

$$\begin{aligned}
P\left(\sup_{s \in (x, 1]} \frac{\mathbb{F}_n(a, s)}{F_0(a, s)} \geq 1 + \frac{n^{-1/2}t}{f_0(a+)}\right) &\leq \exp\left\{-nF_0(a, x)h\left(1 + \frac{n^{-1/2}t}{f_0(a+)}\right)\right\} \\
&= \exp\left\{-\frac{t^2(x-a)}{2f_0(a+)}\psi\left(\frac{n^{-1/2}t}{f_0(a+)}\right)\right\} \\
&\leq \exp\left\{-\frac{t(x-a)}{2} \frac{t/f_0(a+)}{1 + (t/f_0(a+))/(3\sqrt{n})}\right\}.
\end{aligned}$$

Write $u = t/f_0(a+)$ and note that for all $n \geq (k_0/3)^2$ we have

$$\frac{u}{1 + u/(3\sqrt{n})} \geq \frac{u}{1 + u/k_0},$$

which is an increasing function of u . Fix $t_0 > 0$ and let $u_0 = t_0/f_0(a+)$. Then, with $c_0 = u_0/(1 + u_0/k_0) = t_0/(f_0(a+) + t_0/k_0)$ we have that

$$P\left(\widehat{f}_n(x) > f_0(x) + n^{-1/2}t\right) \leq \exp\left\{-c_0 t \frac{x-a}{2}\right\}.$$

We handle the other side in a similar manner.

$$\begin{aligned}
P\left(\widehat{f}_n(x) < f_0(x) - n^{-1/2}t\right) &= P\left(\operatorname{argmax}_{s \in [0,1]} \{\mathbb{F}_n(s, b) - (f_0(b) - n^{-1/2}t)(x - b)\} < x\right) \\
&\leq P\left(\frac{\mathbb{F}_n(s, b)}{F_0(s, b)} \leq 1 - \frac{n^{-1/2}t}{f_0(b)} \text{ for some } s \in [0, x)\right) \\
&\leq P\left(\inf_{s \in [0, x)} \frac{\mathbb{F}_n(s, b)}{F_0(s, b)} \leq 1 - \frac{n^{-1/2}t}{f_0(b)}\right).
\end{aligned}$$

We now bound this using the martingale inequality from Groeneboom et al. (1999, Lemma 2.3).

$$\begin{aligned}
P\left(\inf_{s \in [0, x)} \frac{\mathbb{F}_n(s, b)}{F_0(s, b)} \leq 1 - \frac{n^{-1/2}t}{f_0(b)}\right) &\leq \exp\left\{-nF(x, b)h\left(1 - \frac{n^{-1/2}t}{f_0(b)}\right)\right\} \\
&= \exp\left\{-\frac{t^2(b - x)}{2f_0(b)}\psi\left(-\frac{n^{-1/2}t}{f_0(b)}\right)\right\}.
\end{aligned}$$

Now, note that since \widehat{f}_n is a density, we only consider $t \leq \sqrt{n}f_0(b) = \sqrt{n}f_0(x)$. Therefore, we bound only $\psi(-v)$ for $v \in [0, 1]$, for which we have that $\psi(-v) \geq 1$. Thus it follows that

$$P\left(\widehat{f}_n(x) < f_0(x) - n^{-1/2}t\right) \leq \exp\left\{-\frac{t^2(x'_0 - x)}{2f_0(b)}\right\}.$$

□

Let $(x)_+ = \max(x, 0)$ and $(x)_- = \min(x, 0)$.

Lemma 5.6. *Suppose that f_0 is flat on $(a, b]$ and fix $x \in (a, b)$, and fix $\alpha > 0$. Then, there exists a constant C such that*

$$\begin{aligned}
E\left[\left|\sqrt{n}(\widehat{f}_n(x) - f_0(x))_-\right|^\alpha\right] &\leq C(b - x)^{-\alpha/2}, \\
E\left[\left|\sqrt{n}(\widehat{f}_n(x) - f_0(x))_+\right|^\alpha\right] &\leq C(x - a)^{-\alpha/2},
\end{aligned}$$

with the second bound valid only for $(x - a) \geq \widetilde{c}_0/n$, for some $\widetilde{c}_0 > 0$.

Proof. Using the bounds obtained in Lemma 5.5, we find that

$$\begin{aligned}
E \left[\left| \sqrt{n}(\hat{f}_n(x) - f_0(x))_- \right|^\alpha \right] &= \int_0^\infty \alpha t^{\alpha-1} P(\sqrt{n}(\hat{f}_n(x) - f_0(x))_- > t) dt \\
&= \int_0^{n^{1/2}f_0(x)} \alpha t^{\alpha-1} P(\hat{f}_n(x) < f_0(x) - n^{-1/2}t) dt \\
&\leq \int_0^\infty \alpha t^{\alpha-1} \exp \left\{ -\frac{t^2(b-x)}{2f_0(b)} \right\} dt \\
&= \Gamma(1 + \alpha/2) \left(\frac{2f_0(x_0)}{b-x} \right)^{\alpha/2}.
\end{aligned}$$

For the second inequality, we first fix $t_0 > 0$. We then have

$$\begin{aligned}
E \left[\left| \sqrt{n}(\hat{f}_n(x) - f_0(x))_- \right|^\alpha \right] &= \int_0^\infty \alpha t^{\alpha-1} P(\sqrt{n}(\hat{f}_n(x) - f_0(x))_- > t) dt \\
&= \int_0^{t_0} \alpha t^{\alpha-1} P(\hat{f}_n(x) > f_0(x) + n^{-1/2}t) dt \\
&\quad + \int_{t_0}^\infty \alpha t^{\alpha-1} P(\hat{f}_n(x) > f_0(x) + n^{-1/2}t) dt \\
&\leq t_0^\alpha + \int_{t_0}^\infty \alpha t^{\alpha-1} \exp \left\{ -c_0 \frac{t(x-x_0)}{2} \right\} dt \\
&\leq t_0^\alpha + \Gamma(\alpha + 1) \left(\frac{2}{c_0(x-a)} \right)^\alpha.
\end{aligned}$$

Now, recall that c_0 takes the form $t_0/(f_0(b) + t_0/k_0)$. Therefore, we obtain the bounds

$$\begin{aligned}
\Gamma(\alpha + 1) \left(\frac{2}{c_0(x-a)} \right)^\alpha &\leq 2^\alpha \Gamma(\alpha + 1) (x-a)^{-\alpha} \left(\frac{f_0(b) + t_0/k_0}{t_0} \right)^\alpha \\
&\leq C_\alpha (1 + K)^\alpha \left(\frac{f_0(b)}{x-a} \right)^\alpha t_0^{-\alpha},
\end{aligned}$$

as long as $t_0/k_0 \leq K f_0(x'_0)$ for some choice of K . We optimize the entire quantity in t_0 to find that

$$E \left[\left| \sqrt{n}(\hat{f}_n(x) - f_0(x))_- \right|^\alpha \right] \leq A_\alpha \left(\frac{f_0(b)}{x-a} \right)^{\alpha/2},$$

for some new constant A_α . Now, in order for this optimized bound to hold, we need $t_0 \leq K f_0(b) k_0$, and

$$K^2 f_0(b)^2 k_0^2 \geq C_\alpha (1 + K)^\alpha \left(\frac{f_0(b)}{x-a} \right)^\alpha.$$

The latter translates to $(x-a) \geq \tilde{c}_0 n^{-1}$. □

Proof of Proposition 5.2. The outline of the proof is as follows. We first require pointwise convergence, which follows from Carolan and Dykstra (1999, Theorem 6.4). One can also easily extend this to convergence in finite dimensional distributions. The particular form of the limit follows from the following decomposition of a (time-transformed) Brownian bridge, which is a generalization of Shorack and Wellner (1986, Exercise 2.2.11, page 32). Let F denote any distribution function with compact support, which, without loss of generality, we assume to be $[0, 1]$. Let $a_1 = 0 < b_1 = a_2 < \dots < b_J = 1$. Let $\mathbb{V}, \mathbb{U}_1, \dots, \mathbb{U}_J$ denote independent Brownian bridges. Then

$$\begin{aligned} \mathbb{V}(F(t)) \equiv & \sum_{i=1}^J \left\{ \sum_{j=1}^{i-1} \Delta \mathbb{V}(F(a_i)) + \Delta \mathbb{V}(F(a_i)) \frac{F(t) - F(a_i)}{F(b_i) - F(a_i)} \right. \\ & \left. + \sqrt{F(b_i) - F(a_i)} \mathbb{U}_i \left(\frac{F(t) - F(a_i)}{F(b_i) - F(a_i)} \right) \right\} 1_{(a_i, b_i]}(t) \end{aligned} \quad (5.3)$$

where $\Delta \mathbb{V}(F(a_j)) = \mathbb{V}(F(b_j)) - \mathbb{V}(F(a_j))$.

Recall that the Grenander operator satisfies $\text{gren}(a + bt + ch(t)) = b + c \text{gren}(h(t))$. Also note that F_0 is linear on $(a_i, b_i]$ by assumption. Therefore, from Carolan and Dykstra (1999), the limit of $\sqrt{n}(\hat{f}_n(x) - f_0(x))$ at a point $x \in I_i = (a_i, b_i]$ can be written as

$$\begin{aligned} \text{gren}_{(a_i, b_i]}(\mathbb{V}(F_0(t))) &= \Delta \mathbb{V}(F(a_i)) \frac{1}{b_i - a_i} + \sqrt{p_i} \text{gren} \left(\mathbb{U}_i \left(\frac{t - a_i}{b_i - a_i} \right) \right) \\ &= \frac{1}{b_i - a_i} \left\{ \Delta \mathbb{V}(F(a_i)) + \sqrt{p_i} \text{gren}(\mathbb{U}_i) \left(\frac{t - a_i}{b_i - a_i} \right) \right\}, \end{aligned}$$

from the above characterization. Finally note that $\{\Delta \mathbb{V}(F(a_1)), \dots, \Delta \mathbb{V}(F(a_J))\} \stackrel{d}{=} \{Z_1, \dots, Z_J\}$ as in Proposition 5.2.

The second step is to show that the process

$$\mathbb{S}_n(x) = \sqrt{n}(\hat{f}_n(x) - f_0(x))$$

is tight in $L_\alpha(\mathcal{S})$. For this, we first need a characterization of compact sets in $L_\alpha(\mathcal{S})$ for $\alpha \geq 1$. These appear, for example in Dunford and Schwartz (1958, page 298) (see also Simon (1987)). For \mathcal{S} bounded, a set $K \subset L_\alpha(\mathcal{S})$ is relatively compact if for all $f \in \mathcal{K}$

- (1) $\sup_{f \in K} \int_{\mathcal{S}} |f(x)|^\alpha dx < \infty$,
- (2) $\lim_{\varepsilon \rightarrow 0} \sup_{f \in K} \int_{\mathcal{S}} |f(x + \varepsilon) - f(x)|^\alpha dx \rightarrow 0$.

We then need to show that for all $\varepsilon > 0$, there exists a compact set $K \subset L_\alpha(\mathcal{S})$ such that $\sup_n P(\mathbb{S}_n(\cdot) \in K^c) < \varepsilon$. We will do this by showing that $E[\int |\mathbb{S}_n(x)|^\alpha dx]$ is uniformly bounded in n (for sufficiently large values of n). This clearly establishes condition (1) above. It is also sufficient for condition (2). This is because

\widehat{f}_n is increasing and f_0 is piecewise linear, and therefore, except for a few small regions, $(\widehat{f}_n - f_0)(x)$ is also an increasing function. To bound $E[\int |\mathbb{S}_n(x)|^\alpha dx]$ we use the results of Lemmas 5.4 and 5.6, and consider separately $E[\int |(\mathbb{S}_n(x))_+|^\alpha dx]$ and $E[\int |(\mathbb{S}_n(x))_-|^\alpha dx]$. To simplify the exposition, we write down the details only on a region $(a, b]$ where f_0 is flat. The full result is, of course, immediate.

Thus, the two bounds are as follows:

$$\int_a^b E \left[\left| \sqrt{n}(\widehat{f}_n(x) - f_0(x))_- \right|^\alpha \right] dx \leq C \int_a^b (b-x)^{-\alpha/2} dx,$$

which is finite for $\alpha < 2$. Secondly,

$$\begin{aligned} & \int_a^b E \left[\left| \sqrt{n}(\widehat{f}_n(x) - f_0(x))_+ \right|^\alpha \right] dx \\ & \leq \int_a^{a+\tilde{c}_0/n} E \left[\left| \sqrt{n}(\widehat{f}_n(x) - f_0(x))_+ \right|^\alpha \right] dx + C \int_{a+\tilde{c}_0/n}^b (x-a)^{-\alpha/2} dx \\ & \leq n^{\alpha/2-1} O_p(1) + C \int_a^b (x-a)^{-\alpha/2} dx, \end{aligned}$$

which is again finite for $\alpha < 2$. This completes the proof for $\alpha \in [1, 2)$. \square

Proof of Corollary 5.3. Convergence follows immediately by continuity of the linear functional $\int g(x)\mathbb{S}_n(x)dx$ by Hölder's inequality. We need only check the final form:

$$\begin{aligned} & \int g(x)\mathbb{S}(x)dx \\ & = \sum_{i=1}^J \left\{ Z_i \frac{\int_{a_i}^{b_i} g(x)dx}{b_i - a_i} + \sqrt{p_i} \int_{a_i}^{b_i} \frac{g(x)}{b_i - a_i} \text{gren}(\mathbb{U}_i) \left(\frac{x - a_i}{b_i - a_i} \right) dx \right\} \\ & = \sum_{j=1}^J \left\{ Z_j \frac{\int_{a_j}^{b_j} g(x)dx}{b_j - a_j} + \sqrt{p_j} \int_0^1 g((b_j - a_j)u + a_j) \text{gren}(\mathbb{U}_j)(u) du \right\}. \end{aligned}$$

\square

5.3. Piecewise constant KL density. We next consider the case that $\widehat{f}_0(x)$ can be written in the form given in condition (F1). Let $\mathbb{U}_1, \dots, \mathbb{U}_J$ denote independent standard Brownian bridge processes (each defined on $[0, 1]$), and for each j define \mathbb{U}_j^{mod} as in Remark 2.4 with $I_j = [a_j, b_j]$ replacing $[a, b]$. Also, let $\{Z_1, \dots, Z_J\}$ be an independent multivariate normal with mean zero and covariance $\text{diag}(p) - pp^T$ for $p = (p_1, \dots, p_J)^T$, where $p_j = F_0(b_j) - F_0(a_j) = \widehat{F}_0(b_j) - \widehat{F}_0(a_j)$.

Proposition 5.7. *Suppose that \widehat{f}_0 satisfies conditions (F1) and (F2) with $\mathcal{S}_0 = \mathcal{S}_f$ and that g satisfies condition (G2). Then $\sqrt{n}(\widehat{f}_n(x) - f_0(x))$ converges weakly to*

$\mathbb{S}^{mod}(x)$ in $L_\alpha(\mathcal{S})$ for $\alpha \in (0, 2)$, where

$$\mathbb{S}^{mod}(x) = \sum_{j=1}^J \frac{1}{b_j - a_j} \left\{ Z_j + \sqrt{p_j} \text{gren}(\mathbb{U}_j^{mod}) \left(\frac{x - a_j}{b_j - a_j} \right) \right\} 1_{(a_j, b_j]}(x).$$

Corollary 5.8. *Suppose that \hat{f}_0 satisfies conditions (F1) and (F2) with $\mathcal{S}_0 = \mathcal{S}_f$ and that g satisfies condition (G2). Then $\sqrt{n}(\hat{\mu}_n(g) - \hat{\mu}_0(g)) \Rightarrow Y_J$, where*

$$Y_J^{mod} = \sum_{j=1}^J \left\{ \bar{g}_j Z_j + \sqrt{p_j} \int_0^1 g_j(u) \text{gren}(\mathbb{U}_j^{mod})(u) du \right\}.$$

with $g_j(u)$ and \bar{g}_j defined as in (3.1).

The proof of these results is very close to that of Proposition 5.2, and we omit any details which are the same. The difference lies in the following modifications to Lemmas 5.4 and 5.5. Note that we add the additional requirement that f_0 be bounded below (F2).

Lemma 5.9. *Fix a point $x \in \mathcal{S}_0$ and let $[a, b]$ denote the largest interval I such that $x \in I$ and \hat{f}_0 is constant on I . Then, for all $c > 0$,*

$$\sup_{0 \leq u \leq c/n} \left| \hat{f}_n(a + u) - \hat{f}_0(a + u) \right| = O_p(1).$$

Proof. By the switching relation, it follows that

$$\begin{aligned} & P \left(\hat{f}_n(a + u/n) - \hat{f}_0(a + u/n) < t \right) \\ &= P \left(\hat{f}_n(a + u/n) - \hat{f}_0(b) < t \right) \\ &= P \left(\operatorname{argmax}_{z \in [0, 1]} \left\{ \mathbb{F}_n(z) - (\hat{f}_0(b) + t)z \right\} < a + u/n \right) \\ &= P \left(n \left(\operatorname{argmax}_{z \in [0, 1]} \left\{ \mathbb{F}_n(z) - (\hat{f}_0(b) + t)z \right\} - a \right) < u \right), \end{aligned}$$

and the inner process

$$\begin{aligned} & n \left(\operatorname{argmax}_{z \in [0, 1]} \left\{ \mathbb{F}_n(z) - (\hat{f}_0(b) + t)z \right\} - a \right) \\ &= \operatorname{argmax}_{h \geq -na} \left\{ \mathbb{F}_n(a + h/n) - (t + \hat{f}_0(b))(a + h/n) \right\} = \operatorname{argmax}_{h \geq -na} \{ \mathbb{V}_n(h) \}, \end{aligned}$$

where $\mathbb{V}_n(h) = A_n(h) + B_n(h) - th$, with $A_n(h) = n(\mathbb{F}_n(a + h/n) - \mathbb{F}_n(a)) - n(F_0(a + h/n) - F_0(a))$ and $B_n(h) = n(F_0(a + h/n) - F_0(a)) - \hat{f}_0(b)h$.

Now, the term $\mathbb{N}_n(h) = n(\mathbb{F}_n(a + h/n) - \mathbb{F}_n(a))$ is binomial with mean $n(F_0(a + h/n) - F_0(a)) \rightarrow f_0(a)h$. Therefore, $A_n(h)$ converges to a centred Poisson random variable with mean $f_0(a)$. A similar argument may be used to show convergence

as a process of $A_n(h) \Rightarrow \mathbb{N}(h) - f_0(a)h$, where $\mathbb{N}(\cdot)$ is a Poisson process with rate $\lambda(h) = f_0(a)$. The second piece, $B_n(h)$ satisfies

$$n^{-1}B_n(h) \quad \begin{cases} = 0 & h \in n \{[a, b] \cap \mathcal{W} - a\} \\ < 0 & h \in n \{[a, b] \cap \mathcal{M} - a\} \end{cases}.$$

Thus, if for all $\delta > 0$ $[a, \delta] \cap \mathcal{W} = \{a\}$ then the limit of $B_n(h)$ is 0 if $h = 0$ and is equal to $-\infty$ otherwise (we will call this setting case (A)). If the above assumption is not true (we will call this setting case (B)), then $\lim_{n \rightarrow \infty} B_n(h) = 0$ for all $h \geq 0$. In case (A), it follows that the limit of $\mathbb{V}_n(h)$ is equal to 0 at $h = 0$ and is equal to $-\infty$ otherwise. Therefore, $\arg\max_{h \geq 0} \{\mathbb{V}_n(h)\} = 0$ here. In case (B) the limit of $\mathbb{V}_n(h)$ is a centred (a.k.a. compensated) Poisson process with rate $f_0(a)$. We therefore have that, in case (A),

$$P\left(\widehat{f}_n(a + u/n) - \widehat{f}_0(a + u/n) < t\right) = P\left(\arg\max_{h \geq -na} \{\mathbb{V}_n(h)\} < u\right) \rightarrow 1,$$

and in case (B),

$$\begin{aligned} P\left(\widehat{f}_n(a + u/n) - \widehat{f}_0(a + u/n) < t\right) &= P\left(\arg\max_{h \geq -na} \{\mathbb{V}_n(h)\} < u\right) \\ &\rightarrow P\left(\arg\max_{h \geq 0} \{\mathbb{N}(h) - f_0(a)h - th\} < u\right), \end{aligned}$$

which gives us pointwise convergence in distribution in both cases.

Lastly, note that $\widehat{f}_0(a + u) = \widehat{f}_0(b)$ is a constant, and $\widehat{f}_n(a + u)$ is decreasing in u by definition. Therefore, $\sup_{0 \leq u \leq c/n} |\widehat{f}_n(a + u) - \widehat{f}_0(a + u)| = |\widehat{f}_n(a) - \widehat{f}_0(b)|$, which converges as described above. \square

Lemma 5.10. *Suppose that \widehat{f}_0 is flat on $(a, b]$ and fix $x \in (a, b)$. Assume also that $\inf_{x \in (a, b]} f_0(x) = \alpha_0 > 0$, and let $\widehat{c}_0 = \alpha_0/\widehat{f}_0(b)$. Then, for any $t_0 > 0$ and $k_0 > 0$, there exists a constant $c_0 = t_0/(\widehat{f}_0(b) + t_0/k_0)$ such that*

$$P\left(\widehat{f}_n(x) > \widehat{f}_0(x) + n^{-1/2}t\right) \leq \exp\left\{-\widehat{c}_0 c_0 \frac{t(x - x_0)}{2}\right\} \quad \text{for all } t \geq t_0,$$

for all $n \geq (k_0/3)^2$. Also, for all $t \in [0, \sqrt{n}\widehat{f}_0(x)]$,

$$P\left(\widehat{f}_n(x) < \widehat{f}_0(x) - n^{-1/2}t\right) \leq \exp\left\{-\widehat{c}_0 \frac{t^2(b - x)}{2\widehat{f}_0(x)}\right\},$$

and otherwise the probability is equal to zero.

Proof. Let $\mathbb{F}_n(a, s) = \mathbb{F}_n(s) - \mathbb{F}_n(a)$, and we write $\hat{\theta} = \hat{f}_0(x)$. Repeating the argument for the proof of Lemma 5.5 we obtain that

$$\begin{aligned}
& P\left(\hat{f}_n(x) > \hat{f}_0(x) + n^{-1/2}t\right) \\
& \leq P\left(\frac{\mathbb{F}_n(a, s)}{F_0(a, s)} \geq \frac{(\hat{\theta} + n^{-1/2}t)(s - a)}{F_0(a, s)} \text{ for some } s \in (x, 1]\right) \\
& \leq P\left(\sup_{s \in (x, 1]} \frac{\mathbb{F}_n(a, s)}{F_0(a, s)} \geq \inf_{s \in (x, 1]} \frac{(\hat{\theta} + n^{-1/2}t)(s - a)}{F_0(a, s)}\right) \\
& \leq P\left(\sup_{s \in (x, 1]} \frac{\mathbb{F}_n(a, s)}{F_0(a, s)} \geq \inf_{s \in (x, 1]} \frac{(\hat{\theta} + n^{-1/2}t)(s - a)}{\hat{F}_0(a, s)}\right) \\
& = P\left(\sup_{s \in (x, 1]} \frac{\mathbb{F}_n(a, s)}{F_0(a, s)} \geq 1 + \frac{n^{-1/2}t}{\hat{\theta}}\right)
\end{aligned}$$

since $\hat{F}_0(s) > F_0(s)$ with equality at $s = a, b$. Applying the exponential bounds for binomial variables as before, we find that

$$\begin{aligned}
P\left(\sup_{s \in (x, 1]} \frac{\mathbb{F}_n(a, s)}{F_0(a, s)} \geq 1 + \frac{n^{-1/2}t}{\hat{\theta}}\right) & \leq \exp\left\{-nF_0(a, x)h\left(1 + \frac{n^{-1/2}t}{\hat{\theta}}\right)\right\} \\
& \leq \exp\left\{-\left[\inf_{x \in [a, b]} \frac{f_0(x)}{\hat{\theta}}\right] \frac{t(x - a)}{2} \frac{t/\hat{\theta}}{1 + (t/\hat{\theta})/(3\sqrt{n})}\right\}.
\end{aligned}$$

Therefore, assuming that $\inf_{x \in [a, b]} \frac{f_0(x)}{\hat{\theta}} = \hat{c}_0 > 0$, we can repeat the same argument as for Lemma (5.5).

We handle the other side in a similar manner.

$$\begin{aligned}
& P\left(\hat{f}_n(x) < \hat{f}_0(x) - n^{-1/2}t\right) \\
& \leq P\left(\inf_{s \in [0, x]} \frac{\mathbb{F}_n(s, b)}{F_0(s, b)} \leq \sup_{s \in [0, x]} \frac{(\hat{\theta} - n^{-1/2}t)(s - b)}{F_0(s, b)}\right) \\
& \leq P\left(\inf_{s \in [0, x]} \frac{\mathbb{F}_n(s, b)}{F_0(s, b)} \leq 1 - \frac{n^{-1/2}t}{\hat{\theta}}\right)
\end{aligned}$$

We again bound this using the martingale inequality from Groeneboom et al. (1999, Lemma 2.3).

$$\begin{aligned}
P\left(\inf_{s \in [0, x]} \frac{\mathbb{F}_n(s, b)}{F_0(s, b)} \leq 1 - \frac{n^{-1/2}t}{\hat{\theta}}\right) & \leq \exp\left\{-nF(x, b)h\left(1 - \frac{n^{-1/2}t}{\hat{\theta}}\right)\right\} \\
& = \exp\left\{-\left[\inf_{x \in [a, b]} \frac{f_0(x)}{\hat{\theta}}\right] \frac{t^2(b - x)}{2\hat{\theta}} \psi\left(-\frac{n^{-1/2}t}{\hat{\theta}}\right)\right\}.
\end{aligned}$$

□

5.4. Putting it all together.

Proof of Theorem 3.1. To illustrate the method of proof, we consider a simplified case. Since J is finite in the form of \hat{f}_0 , the proof easily extends to a general setting. Suppose then that $\mathcal{S}_c = [0, a]$ and $\mathcal{S}_f = [a, b]$, so that the support is $\mathcal{S}_0 = [0, b]$. Furthermore, we assume that on \mathcal{S}_f we have $J = 1$. Let $\mathbb{U}_n(x) = \sqrt{n}(\mathbb{F}_n(x) - F_0(x))$. Then

$$\begin{aligned} & \sqrt{n}(\hat{\mu}_n(g) - \hat{\mu}_0(g)) \\ &= \int_0^a g(x) d\mathbb{U}_n(x) + \int_a^b g(x) \sqrt{n} \left(\text{gren}(\mathbb{F}_n)(x) - \hat{f}_0(x) \right) dx + \varepsilon_n, \end{aligned}$$

where $\varepsilon_n = \sqrt{n} \int_0^a g(x) d(\hat{F}_n - \mathbb{F}_n)(x)$. From assumptions (C1) and (G1) it follows that $\varepsilon_n = o_p(1)$ as in Proposition 5.1.

Next, let $\mathbb{W}_n = \mathbb{U}_n(b) - \mathbb{U}_n(a)$, and let $\mathbb{V}_n(x) = \mathbb{U}_n(x) - \mathbb{U}_n(a) - \frac{x-a}{b-a} \mathbb{W}_n$ for $x \in [a, b]$. Lastly, let $\ell(x) = F_0(a) + \hat{f}_0(b)(x - a)$. Then for $x \in (a, b]$,

$$\begin{aligned} \sqrt{n} \left(\text{gren}(\mathbb{F}_n)(x) - \hat{f}_0(x) \right) &= \sqrt{n} \left(\text{gren}(\mathbb{F}_n)(x) - \hat{f}_0(b) \right) \\ &= \text{gren}(\mathbb{U}_n + \sqrt{n}(F_0 - \ell)) \\ &= \frac{1}{b-a} \mathbb{W}_n + \text{gren}(\mathbb{V}_n + \sqrt{n}(F_0 - \ell)), \end{aligned}$$

and we also define $\mathbb{V}_n^{mod} = \mathbb{V}_n + \sqrt{n}(F_0 - \ell)$. We therefore have,

$$\begin{aligned} & \sqrt{n}(\hat{\mu}_n(g) - \hat{\mu}_0(g)) \\ &= \int_0^a g(x) d\mathbb{U}_n(x) + \mathbb{W}_n \frac{1}{b-a} \int_a^b g(x) + \int_a^b g(x) \text{gren}(\mathbb{V}_n^{mod})(x) dx + o_p(1) \\ &= \int_0^b \bar{g}(x) d\mathbb{U}_n(x) + \int_a^b g(x) \text{gren}(\mathbb{V}_n^{mod})(x) dx + o_p(1), \end{aligned}$$

from the definition of \mathbb{W}_n and of \bar{g} . The weak limit of \mathbb{V}_n^{mod} can be established similarly as in Theorem 2.3 and Remark 2.4. The outline of the rest of the proof proceeds as follows:

- (1) Joint weak convergence of $\{\int_0^b \bar{g} d\mathbb{U}_n, \mathbb{V}_n^{mod}(x_1), \dots, \mathbb{V}_n^{mod}(x_k)\}$ to a Gaussian limit.
- (2) Joint weak convergence of $\{\int_0^b \bar{g} d\mathbb{U}_n, \text{gren}(\mathbb{V}_n^{mod})(x_1), \dots, \text{gren}(\mathbb{V}_n^{mod})(x_k)\}$ via the switching relation.
- (3) We have that

$$\text{gren}(\mathbb{V}_n^{mod})(x) = \sqrt{n} \left(\hat{f}_n(x) - \hat{f}_0(x) \right) - \frac{1}{b-a} \mathbb{W}_n,$$

where in Proposition 5.7 we showed that the first term on the right hand side is tight in $L_\alpha(a, b)$. The second term on the right hand side is a tight constant, and therefore $\text{gren}(\mathbb{V}_n^{\text{mod}})(x)$ is also tight in $L_\alpha(a, b)$.

- (4) From (1) and (3) we obtain marginal tightness of the terms $\int_0^b \bar{g} d\mathbb{U}_n$ in \mathbb{R} and $\text{gren}(\mathbb{V}_n^{\text{mod}})(\cdot)$ in $L_\alpha(a, b)$, which implies joint tightness in $\mathbb{R} \times L_\alpha$. The full result now follows by the continuous mapping theorem.

Lastly, we note that since $F_0(z) = \widehat{F}_0(z)$ at $z = a, b$ and \bar{g} is constant on $[a, b]$ then $\int_{S_0} \bar{g}(x) d\mathbb{U}_{F_0}(x) = \int_{S_0} \bar{g}(x) d\mathbb{U}_{\widehat{F}_0}(x)$. \square

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APPENDIX

6.1. Convergence of decreasing densities. All statements in this subsection are taken from Patilea (1997, 2001) and are included for completeness. Related properties for the class of log-concave densities were studied in Cule and Samworth (2010); Dümbgen et al. (2011). Recall that any decreasing density on \mathbb{R}_+ can be written as a mixture of uniform densities

$$f_0(x) = \int_0^\infty \frac{1}{y} 1_{[0,y]}(x) d\mu_0(y),$$

where the mixing probability measure μ_0 is unique. Consider a sequence of decreasing densities f_n , each with a mixing probability measure μ_n . Also define a distance between two densities in \mathcal{F} as

$$\rho(f, g) = \inf_{\varepsilon \geq 0} \{f(x + \varepsilon) \leq g(x) + \varepsilon \ \forall x \geq 0; g(x) - \varepsilon \leq f(x - \varepsilon) \ \forall x \geq 0\},$$

and let $h^2(f, g) = 1/2 \int (\sqrt{f} - \sqrt{g})^2 dx$ denote the square of the Hellinger distance.

Proposition 6.1 (Patilea (1997, Lemma 2.3.1)). *The following are equivalent*

- (1) $h^2(f_n, f_0) \rightarrow 0$,
- (2) $\rho(f_n, f_0) \rightarrow 0$,
- (3) $\mu_n \Rightarrow \mu_0$.

6.2. Additional Proofs.

Proof of Proposition 2.2. (1). Let ℓ denote a general linear function. Then $\widehat{F}_0(x_0) = \min\{\ell(x_0) : \ell \geq F_0\}$. Now consider, $\ell(x)$ the (specific) linear function which passes through the points $(a_0, F_0(a_0)), (b_0, F_0(b_0))$. Then clearly, there is no linear function which always lies above F_0 and strictly below ℓ on (a_0, b_0) . It follows that $\widehat{F}_0(x) = \ell(x)$ on (a_0, b_0) .

Next, suppose that $F_0(a_0) < 1$, with $\widehat{F}_0(a_0) = F_0(a_0)$ but with $\widehat{F}_0(x) > F_0(x)$ for all $x > a_0$. Since F_0 is increasing, it follows that any straight line $\ell(x)$ such that $\ell(a_0) = F_0(a_0)$ and $\ell(x) \geq F_0(x)$ has the form $\ell(x) = F_0(a_0) + m(x - a_0)$ with $m \geq \delta > 0$. But then $\widehat{F}_0(x) \geq F_0(x) + \delta(x - a_0)$, which implies that $\lim_{x \rightarrow \infty} \widehat{F}_0(x) = \infty > 1$, a contradiction to the fact that \widehat{f}_0 is a density.

The last statement follows immediately since $\widehat{F}_0(b_0) - \widehat{F}_0(a_0) = F_0(b_0) - F_0(a_0)$ and \widehat{F}_0 is linear on $(a_0, b_0]$.

(2). Define $\varphi(g) = \int_0^\infty (g(x) - f_0(x))^2 dx$. We first show that the function g minimizes $\varphi(g)$ over \mathcal{D} , the space of decreasing nonnegative functions on \mathbb{R}_+ , if and only if $G(y) = \int_0^y g(x) dx \geq F_0(y)$ for all $y \geq 0$, with equality at all points y such that in the decomposition $g(x) = \int_0^\infty 1_{[0,y]}(x) d\mu_g(y)$, we have $\mu_g(y) > 0$.

The proof of this is a standard argument. We first calculate the directional derivative of $\varphi(g)$, which is $\nabla_b \varphi(g) = \int_0^\infty b(x)(g - f_0)(x) dx$, and recall that any nonnegative decreasing function may be written as $g(x) = \int_0^\infty 1_{[0,y]}(x) d\mu(y)$ for some nonnegative measure μ . Then, if $\varphi(g) \geq \varphi(\widehat{g})$ for all g , it follows that $\nabla_b \varphi(\widehat{g}) \geq 0$ for all b such that $\widehat{g} + \varepsilon b \in \mathcal{D}$ for sufficiently small $\varepsilon > 0$, and this holds for all $b = 1_{[0,y]}$. Also note that if $\mu_{\widehat{g}}(y) > 0$, then $\widehat{g} \pm \varepsilon 1_{[0,y]} \in \mathcal{D}$ for sufficiently small $\varepsilon > 0$. Since $\nabla_{1_{[0,y]}} \varphi(\widehat{g}) = \widehat{G}(y) - F_0(y)$, the result follows.

For the reverse direction, we need to show that if g is such that $G(y) = \int_0^y g(x) dx \geq F_0(y)$ for all $y \geq 0$, with equality at all points y such that in the decomposition $g(x) = \int_0^\infty 1_{[0,y]}(x) d\mu_g(y)$, we have $\mu_g(y) > 0$, then it must have a smaller φ value. Therefore, fix \widehat{g} that satisfies this property, and notice that

$$\begin{aligned} \varphi(g) - \varphi(\widehat{g}) &= \int_0^\infty (g - \widehat{g})^2 dx + 2 \int_0^\infty (g - \widehat{g})(\widehat{g} - f_0) dx \\ &\geq 2 \int_0^\infty \int_0^\infty 1_{[0,y]}(x)(\widehat{g} - f_0)(x) d\mu_g(y) dx \\ &\quad - 2 \int_0^\infty \int_0^\infty 1_{[0,y]}(x)(\widehat{g} - f_0)(x) d\mu_{\widehat{g}}(y) dx \\ &= 2 \int_0^\infty (\widehat{G} - F_0)(y) d\mu_g(y) - 2 \int_0^\infty (\widehat{G} - F_0)(y) d\mu_{\widehat{g}}(y) \geq 0. \end{aligned}$$

The first statement now follows. It remains to show two things: first, that \widehat{G} is the least concave majorant of F_0 and that $\lim_{x \rightarrow \infty} \widehat{G}(x) = 1$, as from the latter it follows

that $\int \hat{g} dx = 1$ and hence minimizing over \mathcal{D} is the same as minimizing over \mathcal{F} . Notice also that the second statement follows from the first, since F_0 is increasing to one. It remains to show that \hat{G} is the least concave majorant of F_0 . By way of contradiction, assume that the concave $\hat{G} \geq F_0$ is not the least concave majorant. Then there exists a concave function \tilde{G} such that $\hat{G} \geq \tilde{G} \geq F_0$, and there exists some interval where $\hat{G}(x) > \tilde{G}(x) \geq F_0(x)$. But then, by the arguments above, \hat{G} cannot be a straight line on this interval, and hence there exists an x such that $\mu_{\hat{g}}(x) > 0$ and $\hat{G}(x) > F_0(x)$. This completes the proof. Note that this could also have been proved using the results of Moreau (1962); see also Hiriart-Urruty and Lemaréchal (2001, pages 46–51).

(3). This follows from the argument in (2), since h is increasing and therefore $\nabla_{(-h)}\varphi(\hat{f}_0) \geq 0$.

(4). This follows as in Robertson et al. (1988, Theorem 1.6.1). Since Φ is convex we have that $\Phi(v) \geq \Phi(u) + \Phi'(u)(v - u)$, for any (sub-)derivative function Φ' . It follows that

$$\int \Phi(f_0 - g_0) dx \geq \int \Phi(\hat{f}_0 - g_0) dx + \int \Phi'(\hat{f}_0 - g_0)(f_0 - \hat{f}_0) dx.$$

The second term is equal to

$$\int_{\mathcal{M}} \Phi'(\hat{f}_0 - g_0)(f_0 - \hat{f}_0) dx = \sum_j \int_{(a_j, b_j]} \Phi'(\hat{f}_0 - g_0)(f_0 - \hat{f}_0) dx,$$

where the intervals $(a_j, b_j]$ denote the sets on which \hat{f}_0 is constant. But here, g_0 is decreasing, and therefore on a fixed $(a_j, b_j]$, $\Phi'(\hat{f}_0 - g_0)$ is increasing. By (3) above it therefore follows that the second term is nonnegative, proving the result.

(5). Let $a = \sup_{x \geq 0} |F_0(x) - G_0(x)|$. Then, by definition, $\hat{F}_0(x) + a \geq G_0(x)$. Also, since $G_0(x) + a$ is concave and always greater than $F_0(x)$, we have that $G_0(x) + a \geq \hat{F}_0(x)$, and the result follows. \square

6.2.1. *Proof of Lemma 4.2.* We note that within this section we make reference to the “bracketing entropy”, in the sense of empirical process theory (van der Vaart and Wellner, 1996), which is different than the entropy functional discussed in Section 4. Define the Bernstein “norm” to be $d_B^2(f, g) = 2 \int (e^{|f-g|} - 1 - |f-g|) dF_0$, and let $h^2(f, g) = 1/2 \int (\sqrt{f} - \sqrt{g})^2 dx$ denote the square of the Hellinger distance.

Lemma 6.2. *Suppose that $f_0/\hat{f}_0 \leq c_0^2$. For $m_f = \log((f + \hat{f}_0)/2\hat{f}_0)$, we have*

- (1) $d_B^2(m_f, m_{\hat{f}_0}) \leq 24c_0^2 h^2(f, \hat{f}_0)$,
- (2) $d_B^2(m_f, m_g) \leq 48c_0^2 h^2(f, g)$.

Proof. For $x \geq -2$, we have $e^{|x|} - 1 - |x| \leq 12(e^{x/2} - 1)^2$. Since $m_f \geq -\log 2$, it follows

$$\begin{aligned} d_B^2(m_f, m_{\hat{f}_0}) &\leq 24 \int (e^{m_f/2} - 1)^2 dF_0 = 24 \int \left(\sqrt{\frac{f + \hat{f}_0}{2\hat{f}_0}} - 1 \right)^2 dF_0 \\ &\leq 48c_0^2 h^2((f + \hat{f}_0)/2, \hat{f}_0) \leq 24c_0^2 h^2(f, \hat{f}_0), \end{aligned}$$

applying van de Geer (2000, Lemma 4.2, page 48) in the last inequality. Now, suppose that $m_f - m_g \geq 0$. Then using a similar argument, we have that $d_B^2(m_f, m_g)$ is bounded above by

$$\begin{aligned} 24 \int \left(\sqrt{\frac{f + \hat{f}_0}{g + \hat{f}_0}} - 1 \right)^2 dF_0 &= 48 \int \left(\sqrt{\frac{f + \hat{f}_0}{2}} - \sqrt{\frac{g + \hat{f}_0}{2}} \right)^2 \frac{f_0}{g + \hat{f}_0} dx \\ &\leq 96c_0^2 h^2\left(\frac{f + \hat{f}_0}{2}, \frac{g + \hat{f}_0}{2}\right) \leq 48c_0^2 h^2(f, g). \end{aligned}$$

If $m_f - m_g \leq 0$, we use instead the bound $e^{|x|} - 1 - |x| \leq 12(e^{|x|/2} - 1)^2$, and obtain

$$d_B^2(m_f, m_g) \leq 48c_0^2 h^2(g, f) = 48c_0^2 h^2(f, g).$$

□

Proof of Lemma 4.2. Let $h_0^2(f, \hat{f}_0) = \frac{1}{2} \int \left(\sqrt{\frac{f}{\hat{f}_0}} - 1 \right)^2 dF_0$ denote the modified Hellinger distance used in Patilea (2001). Since \hat{f}_n is the MLE, $h(x) = \log(x)$ is convex, and $\log x \leq x - 1$,

$$\begin{aligned} 0 &\leq \frac{1}{4} \int \log \frac{\hat{f}_n}{\hat{f}_0} d\mathbb{F}_n \leq \frac{1}{2} \int \log \frac{\hat{f}_n + \hat{f}_0}{2\hat{f}_0} d\mathbb{F}_n \\ &\leq \frac{1}{2} \int m_{\hat{f}_n} d(\mathbb{F}_n - F_0) - \int \left(1 - \sqrt{\frac{\hat{f}_n + \hat{f}_0}{2\hat{f}_0}} \right) dF_0, \end{aligned}$$

where $m_f = \log\left(\frac{f + \hat{f}_0}{2\hat{f}_0}\right)$. Now, from Patilea (2001, Lemma 2.2, (2.2)), we have $\int \left(1 - \sqrt{\frac{f + \hat{f}_0}{2\hat{f}_0}} \right) dF_0 \geq 0$. Hence

$$0 \leq \frac{1}{2} \int \log \frac{\hat{f}_n}{\hat{f}_0} d\mathbb{F}_n \leq \int m_{\hat{f}_n} d(\mathbb{F}_n - F_0),$$

and therefore to prove the lemma it is sufficient to prove that the term on the right hand side is also of order $O_p(n^{-2/3})$. To this end, define $\mathbb{G}_n(m_f) = \sqrt{n} \int m_f(x) d(\mathbb{F}_n - F_0)$, and $\|\mathbb{G}_n\|_{\mathcal{H}} = \sup_{h \in \mathcal{H}} |\mathbb{G}_n(h)|$, and let $\mathcal{M}_{\delta, K} = \{m_f, h_0(f, \hat{f}_0) \leq \delta, f \in \mathcal{F}_K\}$,

with \mathcal{F}_K denoting the class of positive decreasing functions with bounded support and bounded above by K .

Now, by Markov's inequality

$$\begin{aligned}
P(n^{-1/2}\mathbb{G}_n(m_{\hat{f}_n}) \geq \delta_n^2) &\leq P(n^{-1/2}\mathbb{G}_n(m_{\hat{f}_n}) \geq \delta_n^2, m_{\hat{f}_n} \in \mathcal{M}_{\delta_n, K}) + P(h_0(\hat{f}_n, \hat{f}_0) > \delta_n) + P(\hat{f}_n \notin \mathcal{F}_K) \\
&\leq \frac{E[||\mathbb{G}_n||_{\mathcal{M}_{\delta_n, K}}]}{n^{1/2}\delta_n^2} + P(h_0(\hat{f}_n, \hat{f}_0) > \delta_n) + P(\hat{f}_n \notin \mathcal{F}_K). \tag{6.1}
\end{aligned}$$

To finish the proof we will use the results of Patilea (2001) and empirical process theory as in Gao and Wellner (2009).

To obtain the appropriate bracketing entropy bounds on $E[||\mathbb{G}_n||_{\mathcal{M}_{\delta_n}}]$, we will use van der Vaart and Wellner (1996, Lemma 3.4.3, page 324) (we can do this by the first inequality in Lemma 6.2 from which it follows that if $h_0(\hat{f}_n, \hat{f}_0) \leq \delta$ then $d_B(\hat{f}_n, \hat{f}_0) \leq 5\delta$). Next, the results of Lemma 6.2 further show that an L_2 bracket $[\sqrt{f}, \sqrt{g}]$ of densities of size δ leads to a bracket $[m_f, m_g]$ of Bernstein norm size a multiple of δ . Therefore, the bracketing integral of $\mathcal{M}_{\delta, K}$ under the Bernstein norm can be bounded by the bracketing integral of

$$\widetilde{\mathcal{M}}_{\delta, K} = \{f, f \text{ decreasing with } f(0+) \leq K \text{ and support on } [0, A] \text{ and } ||f||_2 \leq \delta\},$$

under the L_2 norm $||f||_2^2 = \int f^2(x)dx$. By Gao and Wellner (2009, Theorem 4), we have that the bracketing integral $J_{[\cdot]}(\delta, \widetilde{\mathcal{M}}_{\delta}, ||\cdot||_2) \leq C(\log K)^{1/4}\delta^{1/2}$, for some constant C and we assume without loss of generality that $A = 1$. Applying van der Vaart and Wellner (1996, Lemma 3.4.3, page 324) it follows that

$$E[||\mathbb{G}_n||_{\mathcal{M}_{\delta_n, K}}] \leq C(\log K)^{1/4}\delta_n^{1/2} \left(1 + (\log K)^{1/4} \frac{\delta_n^{1/2}}{\delta_n^2 \sqrt{n}}\right).$$

We now choose $\delta_n = Mn^{-1/3}$ for $M > 1$ and plug this into (6.1). We obtain

$$\begin{aligned}
P(n^{-1/2}\mathbb{G}_n(m_{\hat{f}_n}) \geq Mn^{-2/3}) &\leq C \frac{(\log K)^{1/4}}{n^{1/6}} \left(1 + \frac{(\log K)^{1/4}}{n^{1/6}}\right) + P(\rho(\hat{f}_n, \hat{f}_0) > Mn^{-1/3}) + P(\hat{f}_n(0+) \geq K). \tag{6.2}
\end{aligned}$$

Now, by definition

$$\hat{f}_0(0+) = \sup_{t>0} \frac{\mathbb{F}_n(t)}{t} = \sup_{t>0} \frac{\mathbb{F}_n(t)}{F_0(t)} \frac{F_0(t)}{t} \leq \sup_{t>0} \frac{F_0(t)}{t} \sup_{t>0} \frac{\mathbb{F}_n(t)}{F_0(t)},$$

and the term $\sup_{t>0} \mathbb{F}_n(t)/F_0(t) = O_p(1)$ (Shorack and Wellner, 1986, Theorem 2, page 345), while $\sup_{t>0} \frac{F_0(t)}{t} = \hat{f}_0(0+)$ which is bounded by assumption. Therefore, $P(\hat{f}_n(0+) \geq K) \leq C/K$, and we can choose $K = n^{1/6}$. Using Patilea (2001, Corollary

5.6 with $\varepsilon = 1$ and $\alpha = 0$) to handle the middle term, we obtain

$$\lim_{M \rightarrow \infty} \limsup_{n \rightarrow \infty} P(n^{-1/2} \mathbb{G}_n(m_{\hat{f}_n}) \geq Mn^{-2/3}) = 0,$$

as required. \square

We finish off this section by commenting on how the conditions of Lemma 4.2 could be relaxed. The conditions that (A) $f_0/\hat{f}_0 < \infty$ and (B) f_0 has bounded support are necessary in our method of proof. However, one could relax the condition that \hat{f}_0 is bounded above and replace it with both

(C) for some $\varepsilon \in (0, 1)$, $\int_{\hat{f}_0} \hat{f}_0^\varepsilon dF_0(x) < \infty$, and

(D) $\limsup_n P(\sup_{t>0} \mathbb{F}_n(t)/t > K_n) = 0$, for $K_n \rightarrow \infty$ such that $\log K_n = o(n^{2/3})$.

The key inequality in (6.3) provides a bound made of up of three terms. The second of these is handled by Patilea (2001, Corollary 5.6), and this continues to hold if (A), (B), and (C) are true. We then have to pick $K = K_n$ such that both the first and third terms go to zero, which is exactly condition (D). Condition (C) is discussed in Patilea (2001). Some ideas on how to achieve (D) can be gained from the results in Balabdaoui et al. (2011).

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